

Demazure crystals and tensor products of perfect Kirillov-Reshetikhin crystals with various levels

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Abstract

In this paper, we study a tensor product of perfect Kirillov-Reshetikhin crystals (KR crystals, for short) whose levels are not necessarily equal. We show that, by tensoring with a certain highest weight element, such a crystal becomes isomorphic as full subgraphs to a certain disjoint union of Demazure crystals contained in a tensor product of highest weight crystals. Moreover, we show that this isomorphism preserves their \mathbb{Z} -gradings, where the \mathbb{Z} -grading on the tensor product of KR crystals is given by the energy function, and that on the other side is given by the minus of the action of the degree operator.

1 Introduction

Crystal bases $B(\Lambda)$ introduced by Kashiwara [13] can be viewed as bases at $q = 0$ of highest weight modules $V(\Lambda)$ of the quantized enveloping algebra $U_q(\mathfrak{g})$ associated with a Kac-Moody Lie algebra \mathfrak{g} . Crystal bases reflect the internal structures of the modules, and are powerful combinatorial tools for studying them.

Crystal bases are also useful for studying certain subspaces of $V(\Lambda)$. For a Weyl group element w , the Demazure module $V_w(\Lambda)$, which is a module of a Borel subalgebra, is defined by the submodule of $V(\Lambda)$ generated by the extremal weight space $V(\Lambda)_{w\Lambda}$. Kashiwara showed in [14] that there exists a certain subset $B_w(\Lambda) \subseteq B(\Lambda)$ which is, in a suitable sense, a crystal basis of $V_w(\Lambda)$. The subset $B_w(\Lambda)$ is called the Demazure crystal. Using the Demazure crystals, he gave a new proof of the character formula for Demazure modules, which expresses the character using the Demazure operators (see [14] or Subsection 4 of the present article).

When \mathfrak{g} is an affine Kac-Moody Lie algebra, there is another class of modules having crystal bases called Kirillov-Reshetikhin modules $W^{r,s}$ (KR modules for short), where r is a node in the classical Dynkin diagram and s is a positive integer. KR modules are finite-dimensional irreducible $U'_q(\mathfrak{g})$ -modules, where $U'_q(\mathfrak{g})$ is a quantum affine algebra without the degree operator. At least when \mathfrak{g} is nonexceptional, it was proved that every $W^{r,s}$ has a crystal basis $B^{r,s}$ [12, 24, 25], which is called the Kirillov-Reshetikhin (KR) crystal.

Demazure crystals and KR crystals are known to have strong relations, and the study of the relationship between them has been the subject of many articles. For example, see [4, 16, 18, 19, 22, 28].

Among these articles, [28] by Schilling and Tingley is quite important for the present article. In the article, they studied a tensor product of perfect

KR crystals for nonexceptional \mathfrak{g} whose levels are all the same (perfectness is a technical condition for a finite $U'_q(\mathfrak{g})$ -crystal which allows one to use the crystal to construct highest weight crystals, see [11] or Subsection 5.2 of the present article). They proved that, by tensoring with a certain highest weight element, such a crystal becomes isomorphic to a certain Demazure crystal as full subgraphs. Moreover, they also showed that this isomorphism preserves their \mathbb{Z} -gradings. Here, the tensor product of perfect KR crystals are \mathbb{Z} -graded by the energy function, which is a certain \mathbb{Z} -function defined in a combinatorial way, and the \mathbb{Z} -grading of Demazure crystal is given by the minus of the action of the degree operator. Since the Demazure crystal has a character formula as stated above, these results imply that the weight sum with the energy function of the tensor product of perfect KR crystals (with same levels) can be expressed by the Demazure character formula.

The aim of this article is to generalize the above results to a tensor product of perfect KR crystals whose levels are not necessarily equal. In this case, the tensor product of perfect KR crystals, tensored with a highest weight element, is no longer isomorphic to a single Demazure crystal. We show in this article, however, that it is isomorphic to a certain disjoint union of Demazure crystals contained in a tensor product of $B(\Lambda)$'s, and that this isomorphism also preserves their \mathbb{Z} -gradings.

Before stating our results, we prepare some notation. For a crystal B and a Dynkin diagram automorphism τ , we define a new crystal $\tilde{\tau}(B) = \{\tilde{\tau}(b) \mid b \in B\}$ whose weight function is $\text{wt}(\tilde{\tau}(b)) = \tau(\text{wt}(b))$ and Kashiwara operators are

$$e_i \tilde{\tau}(b) = \tilde{\tau}(e_{\tau^{-1}(i)} b), \quad f_i \tilde{\tau}(b) = \tilde{\tau}(f_{\tau^{-1}(i)} b).$$

Let S be a subset of a crystal B , w a Weyl group element with a reduced expression $w = s_{i_k} \cdots s_{i_1}$ and τ a Dynkin automorphism. We denote by $\mathcal{F}_{w\tau}(S)$ the subset of $\tilde{\tau}(B)$ defined by

$$\mathcal{F}_{w\tau}(S) = \bigcup_{j_1, \dots, j_k \geq 0} f_{i_k}^{j_1} \cdots f_{i_1}^{j_k} \tilde{\tau}(S).$$

(All the subsets $\mathcal{F}_{w\tau}(S)$ appearing in this article do not depend on the choices of the expressions of w .) For a dominant integral weight Λ , we denote by u_Λ the highest weight element of $B(\Lambda)$.

Now we state our results. Assume that \mathfrak{g} is a nonexceptional affine Kac-Moody Lie algebra, and let $B^{r_1, c_{r_1} s_1}, \dots, B^{r_p, c_{r_p} s_p}$ be perfect KR crystals. Here, c_r is a particular constant which ensures the perfectness for the KR crystal $B^{r, c_r s}$. We assume $s_1 \leq s_2 \leq \cdots \leq s_p$, and put $\mu_j = c_{r_j} w_0(\varpi_{r_j})$ for $1 \leq j \leq p$, where w_0 is the longest element of the Weyl group of the simple Lie subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$ whose simple roots are the nodes of the classical Dynkin diagram, and ϖ_{r_j} is the fundamental weight of \mathfrak{g}_0 . Then the following theorem is proved, which is the main theorem of the present paper (Theorem 7.1):

Theorem 1.1. *There exists an isomorphism*

$$u_{s_p \Lambda_0} \otimes B^{r_p, c_{r_p} s_p} \otimes \cdots \otimes B^{r_1, c_{r_1} s_1} \\ \xrightarrow{\sim} \mathcal{F}_{t_{\mu_p}}(u_{(s_p - s_{p-1}) \Lambda_0} \otimes \cdots \otimes \mathcal{F}_{t_{\mu_2}}(u_{(s_2 - s_1) \Lambda_0} \otimes \mathcal{F}_{t_{\mu_1}}(u_{s_1 \Lambda_0})) \cdots)$$

of full subgraphs, where t_μ denotes the translation and Λ_0 denotes the fundamental weight of \mathfrak{g} . Moreover, this isomorphism preserves the \mathbb{Z} -gradings up to a shift.

Using the combinatorial excellent filtration theorem [20, 8], it is easy to see that the right hand side of the above isomorphism is a disjoint union of Demazure crystals.

Let $B = B^{r_p, c_{r_p} s_p} \otimes \dots \otimes B^{r_1, c_{r_1} s_1}$. Similarly as a Demazure crystal, the weight sum of the right hand side also can be expressed using Demazure operators. Hence we obtain the following corollary (Corollary 7.2):

Corollary 1.2. *Let $\text{aff} : P_{\text{cl}} \rightarrow P$ denote the canonical section of the projection from affine weight lattice P to the classical weight lattice P_{cl} . Then we have*

$$\begin{aligned} e^{s_p \Lambda_0 + C\delta} \sum_{b \in B} e^{\text{aff} \circ \text{wt}(b) - \delta D(b)} \\ = D_{t_{\mu_p}} \left(e^{(s_p - s_{p-1}) \Lambda_0} \dots D_{t_{\mu_2}} \left(e^{(s_2 - s_1) \Lambda_0} \cdot D_{t_{\mu_1}} (e^{s_1 \Lambda_0}) \right) \dots \right) \end{aligned}$$

for some constant C , where D_{t_μ} is the Demazure operator associated with t_μ (see Section 4).

Let $X(B, \mu, q)$ denote the one-dimensional configuration sum [5, 6] associated with the crystal B and a dominant integral weight μ of \mathfrak{g}_0 . Then the above corollary is equivalent to the following (Corollary 7.3):

Corollary 1.3. *Let P_0^+ be the set of dominant integral weights of \mathfrak{g}_0 and $\text{ch } V_{\mathfrak{g}_0}(\mu)$ the character of the irreducible module of \mathfrak{g}_0 with highest weight μ . Then we have*

$$\begin{aligned} q^{-C} \sum_{\mu \in P_0^+} X(B, \mu, q) \text{ch } V_{\mathfrak{g}_0}(\mu) \\ = e^{-s_p \Lambda_0} D_{t_{\mu_p}} \left(e^{(s_p - s_{p-1}) \Lambda_0} \dots D_{t_{\mu_2}} \left(e^{(s_2 - s_1) \Lambda_0} \cdot D_{t_{\mu_1}} (e^{s_1 \Lambda_0}) \right) \dots \right), \end{aligned}$$

where we set $q = e^{-\delta}$.

Corollary 1.3 has an important application (and in fact this is the main motivation of this work). The $X = M$ conjecture presented in [5, 6] asserts that a one-dimensional configuration sum is equal to a fermionic form which is defined as a generating function of some combinatorial objects called rigged configurations. In [23], it is proved that when \mathfrak{g} is of type $A_n^{(1)}, D_n^{(1)}$ or $E_n^{(1)}$, the fermionic forms also satisfy a similar equation as in Corollary 1.3. Then we can prove the $X = M$ conjecture in the cases $\mathfrak{g} = A_n^{(1)}, D_n^{(1)}$ from these equations (for details see [23]).

The plan of this article is as follows. In Section 2, we fix basic notation used in the article. In Section 3, we briefly review the definition of crystals, and in Section 4, we review the results on Demazure crystals. In Section 5, we review the results on KR crystals, and construct the isomorphism in Theorem 1.1. In Section 6, we review the definition and some results on the energy functions, and finally in Section 7, we show that the isomorphism constructed in Section 5 preserves the \mathbb{Z} -gradings, which completes the proof of Theorem 1.1.

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2 Notation and basics

2.1 Affine Kac-Moody Lie algebra

Let \mathfrak{g} be a complex affine Kac-Moody Lie algebra with Cartan subalgebra \mathfrak{h} , Dynkin node set $I = \{0, \dots, n\}$, Dynkin diagram Γ and Cartan matrix $A = (a_{ij})_{i,j \in I}$. In this article, we use the Kac's labeling of nodes of Dynkin diagrams in [9, Section 4.8]. Let $\alpha_i \in \mathfrak{h}^*$ and $\alpha_i^\vee \in \mathfrak{h}$ ($i \in I$) be the simple roots and the simple coroots respectively, and $\Delta \subseteq \mathfrak{h}^*$ the root system of \mathfrak{g} . Let (a_0, \dots, a_n) (resp. $(a_0^\vee, \dots, a_n^\vee)$) be the unique sequence of relatively prime positive integers satisfying

$$\sum_{j \in I} a_{ij} a_j = 0 \text{ for all } i \in I \text{ (resp. } \sum_{i \in I} a_i^\vee a_{ij} = 0 \text{ for all } j \in I).$$

Let $d \in \mathfrak{h}$ be the degree operator, which is any element satisfying $\langle \alpha_i, d \rangle = \delta_{0i}$ for $i \in I$, $K = \sum_{i \in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h}$ the canonical central element, $\delta = \sum_{i \in I} a_i \alpha_i \in \mathfrak{h}^*$ the null root, and W the Weyl group of \mathfrak{g} with simple reflections $\{s_i \mid i \in I\}$. In this article, we fix an arbitrary positive integer N , and define the weight lattice of \mathfrak{g} by

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \ (i \in I), \langle \lambda, d \rangle \in N^{-1}\mathbb{Z}\}. \quad (1)$$

(In the next subsection, we impose some condition on N so that P is preserved by the action of the extended affine Weyl group \widetilde{W} .) Put $P^+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \ (i \in I)\}$, and let $\Lambda_i \in P^+$ ($i \in I$) be any element satisfying

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij} \text{ for } j \in I.$$

(Note that we do not assume $\langle \Lambda_i, d \rangle = 0$.) Then we have $P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i + N^{-1}\mathbb{Z}\delta$. For $\lambda \in P$, we call the integer $\langle \lambda, K \rangle$ the *level* of λ , and for $\ell \in \mathbb{Z}$ we denote $P^\ell = \{\lambda \in P \mid \langle \lambda, K \rangle = \ell\}$. Let $(\ , \)$ be a W -invariant symmetric bilinear form on \mathfrak{h}^* satisfying

$$(\alpha_i, \alpha_j) = a_i^\vee a_j^{-1} a_{ij} \text{ for } i, j \in I, \quad (\alpha_i, \Lambda_0) = \delta_{0i} a_0^{-1} \text{ for } i \in I.$$

Let $\text{cl}: \mathfrak{h}^* \rightarrow \mathfrak{h}^*/\mathbb{C}\delta$ be the canonical projection, and put $P_{\text{cl}} = \text{cl}(P)$, $P_{\text{cl}}^+ = \text{cl}(P^+)$, $P_{\text{cl}}^\ell = \text{cl}(P^\ell)$ for $\ell \in \mathbb{Z}$ and $(P_{\text{cl}}^+)^{\ell} = P_{\text{cl}}^+ \cap P_{\text{cl}}^\ell$. Since W fixes δ , W acts on $\mathfrak{h}^*/\mathbb{C}\delta$ and P_{cl} . For $i \in I$, define $\varpi_i \in P_{\text{cl}}^0$ by $\varpi_i = \text{cl}(\Lambda_i) - a_i^\vee \text{cl}(\Lambda_0)$. Note that $\varpi_0 = 0$ and ϖ_i for $i \in I \setminus \{0\}$ satisfies

$$\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij} \text{ for } j \in I \setminus \{0\}, \quad \langle \varpi_i, \alpha_0^\vee \rangle = -a_i^\vee.$$

We define $\text{aff}: \mathfrak{h}^*/\mathbb{C}\delta \rightarrow \mathfrak{h}^*$ by the unique section of cl satisfying $\langle \text{aff}(\lambda), d \rangle = 0$ for all $\lambda \in \mathfrak{h}^*/\mathbb{C}\delta$. When no confusion is possible, we omit the notation $\text{cl}(\ast)$ for simplicity. In particular, we often write $\text{cl}(\Lambda_i)$ and $\text{cl}(\alpha_i)$ simply as Λ_i and α_i .

Let $I_0 = I \setminus \{0\}$ and $\mathfrak{g}_0 \subseteq \mathfrak{g}$ the simple Lie subalgebra whose Dynkin node set is I_0 with Cartan subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{h}$ and Weyl group $W_0 \subseteq W$. Let $\varpi_j^\vee \in \mathfrak{h}_0$ ($j \in I_0$) be the unique element satisfying

$$\langle \alpha_i, \varpi_j^\vee \rangle = \delta_{ij} \text{ for } i \in I_0,$$

which also satisfies

$$\langle \alpha_0, \varpi_j^\vee \rangle = -a_j/a_0. \quad (2)$$

For the notational convenience, we put $\varpi_0^\vee = 0$. Denote by w_0 the longest element of W_0 . We canonically identify P_{cl}^0 with the weight lattice of \mathfrak{g}_0 .

The bilinear form $(\ , \)$ induces a bilinear form on P_{cl}^0 , which is also denoted by $(\ , \)$. Then we have

$$(\lambda, \varpi_i) = a_i^\vee a_i^{-1} \langle \lambda, \varpi_i^\vee \rangle \quad (3)$$

for $i \in I$ and $\lambda \in P_{\text{cl}}^0$.

2.2 Dynkin automorphisms and the extended affine Weyl group

Let $\text{Aut}(\Gamma)$ be the group of automorphisms of the Dynkin diagram Γ , that is, the group of permutations τ of I satisfying $a_{ij} = a_{\tau(i)\tau(j)}$ for all $i, j \in I$. Then $\tau \in \text{Aut}(\Gamma)$ satisfies

$$a_{\tau(i)} = a_i \text{ and } a_{\tau(i)}^\vee = a_i^\vee \text{ for all } i \in I. \quad (4)$$

As [9, (6.5.2)], we define for $\lambda \in P_{\text{cl}}^0$ an endomorphism t_λ of \mathfrak{h}^* by

$$t_\lambda(\mu) = \mu + \langle \mu, K \rangle \text{aff}(\lambda) - \left((\mu, \text{aff}(\lambda)) + \frac{1}{2} (\text{aff}(\lambda), \text{aff}(\lambda)) \langle \mu, K \rangle \right) \delta. \quad (5)$$

The map $\lambda \mapsto t_\lambda$ defines an injective group homomorphism from P_{cl}^0 to the group of automorphisms of \mathfrak{h}^* orthogonal with respect to $(\ , \)$.

Let $c_i = \max(1, a_i/a_i^\vee)$ for $i \in I_0$, and define sublattices M and \widetilde{M} of P_{cl}^0 by

$$M = \sum_{w \in W_0} \mathbb{Z} w \cdot \text{cl}(\alpha_0/a_0), \quad \widetilde{M} = \bigoplus_{i \in I_0} \mathbb{Z} c_i \varpi_i.$$

Let $T(M)$ and $T(\widetilde{M})$ be the subgroups of $\text{GL}(\mathfrak{h}^*)$ defined by

$$T(M) = \{t_\lambda \mid \lambda \in M\}, \quad T(\widetilde{M}) = \{t_\lambda \mid \lambda \in \widetilde{M}\}.$$

It is known that [9, Proposition 6.5]

$$W \cong W_0 \ltimes T(M).$$

Define the subgroup \widetilde{W} of $\text{GL}(\mathfrak{h}^*)$ by

$$\widetilde{W} = W_0 \ltimes T(\widetilde{M}),$$

which is called the *extended affine Weyl group*. The action of \widetilde{W} preserves Δ , and elements $w \in W_0$ and $\lambda \in \widetilde{M}$ satisfy

$$wt_\lambda w^{-1} = t_{w(\lambda)}.$$

In the sequel, we assume that the positive integer N in (1) satisfies

$$(\text{aff}(\lambda), \text{aff}(\lambda))/2 \in N^{-1}\mathbb{Z} \text{ for all } \lambda \in \widetilde{M}, \quad (6)$$

which ensures that \widetilde{W} preserves P .

Let $C \subseteq \mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} P$ be the fundamental chamber (i.e. $C = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda, \alpha_i) \geq 0 \text{ for all } i \in I\}$), and let $\Sigma \subseteq \widetilde{W}$ be the subgroup consisting of elements preserving C . Then we have

$$\widetilde{W} \cong W \rtimes \Sigma.$$

Note that an element $w \in \widetilde{W}$ belongs to Σ if and only if w preserves the set of simple roots $\{\alpha_0, \dots, \alpha_n\}$. Hence $\tau \in \Sigma$ induces a permutation of I (also denoted by τ) by $\tau(\alpha_i) = \alpha_{\tau(i)}$, which belongs to $\text{Aut}(\Gamma)$ since (\cdot, \cdot) is τ -invariant. By abuse of notation, we denote by Σ both the subgroup of \widetilde{W} and the subgroup of $\text{Aut}(\Gamma)$.

We describe $\Sigma \subseteq \text{Aut}(\Gamma)$ explicitly. A node $i \in I$ is called a *special* node if $i \in \text{Aut}(\Gamma) \cdot 0$. Let $I^s \subseteq I$ be the set of special nodes. I^s for nonexceptional \mathfrak{g} are as follows:

$$\begin{cases} \{0, 1, \dots, n\} & \text{for } A_n^{(1)}, \\ \{0, 1\} & \text{for } B_n^{(1)}, A_{2n-1}^{(2)}, \\ \{0, n\} & \text{for } C_n^{(1)}, D_{n+1}^{(2)}, \\ \{0, 1, n-1, n\} & \text{for } D_n^{(1)}, \\ \{0\} & \text{for } A_{2n}^{(2)}. \end{cases}$$

Assume $i \in I^s \setminus \{0\}$ (in particular $\mathfrak{g} \neq A_{2n}^{(2)}$), and define $\tau^i \in \widetilde{W}$ by

$$\tau^i = t_{\varpi_i} w_i,$$

where w_i denotes the unique element of W_0 which maps the simple system $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of \mathfrak{g}_0 to $\{-\theta, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n\}$ with $\theta = \delta - \alpha_0 \in \Delta$. For $i = 0$, we put $\tau^0 = \text{id}$. The following proposition is well-known, but we give the proof for completeness:

Proposition 2.1. (i) For all $i \in I^s$, τ^i belongs to Σ .
(ii) The map $I^s \rightarrow \Sigma$ defined by $i \mapsto \tau^i$ is bijective.
(iii) If $\tau \in \Sigma$ satisfies $\tau(i) = 0$, then we have $\tau = (\tau^i)^{-1}$.

Proof. If \mathfrak{g} is of type $A_{2n}^{(2)}$, then $I^s = \{0\}$ and $\widetilde{M} = M$, which obviously imply the assertions. So, assume \mathfrak{g} is not of this type. (i) Recall that w_i maps $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ to $\{-\theta, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n\}$. Then $w_i(-\theta) = \alpha_i$ also holds, and hence it is easily checked from the equation (5) that $\tau^i = t_{\varpi_i} w_i$ preserves the set $\{\alpha_0, \dots, \alpha_n\}$, which implies τ^i belongs to Σ . (ii) The injectivity is obvious. Let $\tau \in \Sigma \setminus \{\text{id}\}$ be an arbitrary element, and decompose it as $\tau = t_{\lambda_\tau} w_\tau$ where $\lambda_\tau \in \widetilde{M}$ and $w_\tau \in W_0$. Since t_{λ_τ} acts trivially on P_{cl}^0 , we have $w_\tau(\text{cl}(\alpha_j)) = \text{cl}(\alpha_{\tau(j)})$ for $j \in I$, which implies $w_\tau = w_{\tau(0)}$. Then since

$$t_{\lambda_\tau}(\alpha_j) = \tau w_\tau^{-1}(\alpha_j) = \alpha_j - \delta_{j, \tau(0)} \delta \quad \text{for } j \in I_0,$$

(5) forces $\lambda_\tau = \varpi_{\tau(0)}$, and the surjectivity follows. From the proof of (ii), we see that $\tau(0) = i$ implies $\tau = \tau^i$. Hence, the assertion (iii) follows. \square

Here, we describe τ^i for $i \in I^s \setminus \{0\}$ explicitly.

$A_n^{(1)}$: $\tau^i(j) = j + i \bmod n + 1$ for all $j \in I$.
 $B_n^{(1)}, D_{n+1}^{(2)}$: $\tau^1 = (0, 1)$.
 $C_n^{(1)}, A_{2n-1}^{(2)}$: $\tau^n(j) = n - j$ for all $j \in I$.

$D_n^{(1)}$, n odd: $\tau^1 = (0, 1)(n-1, n)$,
 $\tau^{n-1}(0, 1, n-1, n) = (n-1, n, 1, 0)$, $\tau^{n-1}(j) = n-j$ for $j \in I \setminus I^s$,
 $\tau^n(0, 1, n-1, n) = (n, n-1, 0, 1)$, $\tau^n(j) = n-j$ for $j \in I \setminus I^s$. $D_n^{(1)}$, n even:
 $\tau^1 = (0, 1)(n-1, n)$,
 $\tau^{n-1}(0, 1, n-1, n) = (n-1, n, 0, 1)$, $\tau^{n-1}(j) = n-j$ for $j \in I \setminus I^s$,
 $\tau^n(0, 1, n-1, n) = (n, n-1, 1, 0)$, $\tau^n(j) = n-j$ for $j \in I \setminus I^s$.

In the sequel, we assume that the fundamental weights $\Lambda_0, \dots, \Lambda_n$ are chosen to satisfy $\tau(\Lambda_j) = \Lambda_{\tau(j)}$ for all $\tau \in \Sigma$ and $j \in I$. This is always possible by choosing Λ_j arbitrarily for representatives of I/Σ and setting $\Lambda_{\tau(j)} = \tau(\Lambda_j)$ for $\tau \in \Sigma$. Then each element $\tau \in \text{Aut}(\Gamma)$ acts on P by $\tau(\Lambda_i) = \Lambda_{\tau(i)}$ and $\tau(\delta) = \delta$.

3 Definition of crystals

Let $U_q(\mathfrak{g})$ be the quantum affine algebra associated with \mathfrak{g} and $U'_q(\mathfrak{g})$ the one without the degree operator. The weight lattices of $U_q(\mathfrak{g})$ and $U'_q(\mathfrak{g})$ are P and P_{cl} respectively.

A $U_q(\mathfrak{g})$ -crystal (resp. $U'_q(\mathfrak{g})$ -crystal) is by definition a set B equipped with weight function $\text{wt}: B \rightarrow P$ (resp. $\text{wt}: B \rightarrow P_{\text{cl}}$) and Kashiwara operators $e_i, f_i: B \rightarrow B \sqcup \{0\}$ for $i \in I$ satisfying

$$\begin{aligned}
 \text{wt}(e_i b) &= \text{wt}(b) + \alpha_i \text{ and } f_i(e_i b) = b \quad \text{for all } i \in I, b \in B \text{ such that } e_i b \neq 0, \\
 \text{wt}(f_i b) &= \text{wt}(b) - \alpha_i \text{ and } e_i(f_i b) = b \quad \text{for all } i \in I, b \in B \text{ such that } f_i b \neq 0,
 \end{aligned}$$

and $\langle \text{wt}(b), \alpha_i^\vee \rangle = \varphi_i(b) - \varepsilon_i(b)$ where

$$\varepsilon_i(b) = \max\{k \geq 0 \mid e_i^k b \neq 0\}, \quad \varphi_i(b) = \max\{k \geq 0 \mid f_i^k b \neq 0\}.$$

In this article, we always assume that $\varepsilon_i(b) < \infty$ and $\varphi_i(b) < \infty$. We call B a crystal if B is either a $U_q(\mathfrak{g})$ -crystal or a $U'_q(\mathfrak{g})$ -crystal.

Remark 3.1. A $U_q(\mathfrak{g})$ -crystal B can be regarded naturally as a $U'_q(\mathfrak{g})$ -crystal by replacing the weight function $\text{wt}: B \rightarrow P$ by $\text{cl} \circ \text{wt}: B \rightarrow P_{\text{cl}}$.

For two crystals B_1 and B_2 , their tensor product $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ is defined with weight function $\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2)$ and Kashiwara operators

$$\begin{aligned}
 e_i(b_1 \otimes b_2) &= \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
 f_i(b_1 \otimes b_2) &= \begin{cases} f_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}
 \end{aligned}$$

For crystals B_1, B_2 and their subsets $S_1 \subseteq B_1, S_2 \subseteq B_2$, we say S_1 and S_2 are *isomorphic as full subgraphs* and write $S_1 \cong S_2$ if there exists a bijection Ψ from $S_1 \sqcup \{0\}$ to $S_2 \sqcup \{0\}$ satisfying $\Psi(0) = 0$, $\text{wt}\Psi(b) = \text{wt}(b)$ for $b \in S_1$, $\Psi(e_i b) = e_i \Psi(b)$ for $b \in S_1$ such that $e_i b \in S_1 \sqcup \{0\}$, and $\Psi(f_i b) = f_i \Psi(b)$ for $b \in S_1$ such that $f_i b \in S_1 \sqcup \{0\}$.

For a crystal B and $\tau \in \text{Aut}(\Gamma)$, we define a crystal $\tilde{\tau}(B)$ as follows: as a set $\tilde{\tau}(B) = \{\tilde{\tau}(b) \mid b \in B\} \cong B$. Its weight function and Kashiwara operators are defined by

$$\begin{aligned} \text{wt}(\tilde{\tau}(b)) &= \tau(\text{wt}(b)) \text{ and} \\ e_i \tilde{\tau}(b) &= \tilde{\tau}(e_{\tau^{-1}(i)} b), \quad f_i \tilde{\tau}(b) = \tilde{\tau}(f_{\tau^{-1}(i)} b), \end{aligned} \tag{7}$$

where $\tilde{\tau}(0)$ is understood as 0. Obviously we have

$$\tilde{\tau}(B_1 \otimes B_2) \cong \tilde{\tau}(B_1) \otimes \tilde{\tau}(B_2)$$

for two crystals B_1 and B_2 . For a subset $S \subseteq B$, a subset $\tilde{\tau}(S) \subseteq \tilde{\tau}(B)$ is defined in the obvious way.

For $J \subseteq I$, we denote by $U_q(\mathfrak{g}_J)$ the subalgebra of $U_q(\mathfrak{g})$ whose simple roots are J . If $J = I_0$, we write $U_q(\mathfrak{g}_0) = U_q(\mathfrak{g}_{I_0})$. $U_q(\mathfrak{g}_J)$ -crystals are defined in a similar way. For a crystal B and a proper subset J of I , a connected component of B regarded as a $U_q(\mathfrak{g}_J)$ -crystal is called a $U_q(\mathfrak{g}_J)$ -component of B .

Definition 3.2 ([1]). We say a crystal B is *regular* if for every proper subset J of I , B regarded as a $U_q(\mathfrak{g}_J)$ -crystal is isomorphic to a direct sum of the crystal bases of integrable highest weight $U_q(\mathfrak{g}_J)$ -modules.

Let $J \subseteq I$. For a crystal B , we say that $b \in B$ is $U_q(\mathfrak{g}_J)$ -highest weight if $e_j b = 0$ for all $j \in J$. For a proper subset J of I and a regular crystal B , every $U_q(\mathfrak{g}_J)$ -component of B contains a unique $U_q(\mathfrak{g}_J)$ -highest weight element.

By [15], the actions of simple reflections on a regular crystal B defined by

$$S_{s_i}(b) = \begin{cases} f_i^{\langle \text{wt}(b), \alpha_i^\vee \rangle} b & \text{if } \langle \text{wt}(b), \alpha_i^\vee \rangle \geq 0, \\ e_i^{-\langle \text{wt}(b), \alpha_i^\vee \rangle} b & \text{if } \langle \text{wt}(b), \alpha_i^\vee \rangle < 0 \end{cases}$$

are extended to the action of W denoted by $w \mapsto S_w$. For every $w \in W$ and $b \in B$, we have $\text{wt}(S_w(b)) = w(\text{wt}(b))$.

4 Demazure crystals

For a subset S of a crystal B and $i \in I$, we define $\mathcal{F}_i S = \{f_i^k b \mid b \in S, k \geq 0\} \setminus \{0\} \subseteq B$.

For $\Lambda \in P^+$, let $V(\Lambda)$ denote the integrable highest weight $U_q(\mathfrak{g})$ -module with highest weight Λ , and $B(\Lambda)$ its crystal basis with highest weight element u_Λ . Let w be an element of W and $w = s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$ its reduced expression. Then it is known that the subset

$$B_w(\Lambda) = \mathcal{F}_{i_k} \mathcal{F}_{i_{k-1}} \cdots \mathcal{F}_{i_1} \{u_\Lambda\} \subseteq B(\Lambda)$$

is independent of the choice of the reduced expression of w [14].

Definition 4.1. The subset $B_w(\Lambda)$ of $B(\Lambda)$ is called the *Demazure crystal* associated with Λ and w .

Remark 4.2. Let \mathfrak{b} be a Borel subalgebra of \mathfrak{g} and $U_q(\mathfrak{b}) \subseteq U_q(\mathfrak{g})$ the corresponding quantized enveloping algebra. The *Demazure module* $V_w(\Lambda)$ is defined by the $U_q(\mathfrak{b})$ -submodule of $V(\Lambda)$ generated by the weight space $V(\Lambda)_{w(\Lambda)}$. The Demazure crystal $B_w(\Lambda)$ is known to be, in a suitable sense, the crystal basis of $V_w(\Lambda)$ [14], which is why it is named Demazure crystal.

For a subset S of a crystal and $w \in W$ with a reduced expression $w = s_{i_k} \cdots s_{i_1}$, we write $\mathcal{F}_w S = \mathcal{F}_{i_k} \cdots \mathcal{F}_{i_1} S$ if it is well-defined. For example, $\mathcal{F}_w \{u_\Lambda\} = B_w(\Lambda)$.

Lemma 4.3. *Let $\Lambda \in P^+$ and $w \in W$.*

- (i) $\tilde{\tau} B_w(\Lambda) \cong B_{\tau w \tau^{-1}}(\tau(\Lambda))$ for $\tau \in \text{Aut}(\Gamma)$.
- (ii) For $i \in I$, we have

$$\mathcal{F}_i B_w(\Lambda) = \begin{cases} B_w(\Lambda) & \text{if } \ell(s_i w) = \ell(w) - 1, \\ B_{s_i w}(\Lambda) & \text{if } \ell(s_i w) = \ell(w) + 1, \end{cases} \quad (8)$$

where ℓ denotes the length function.

- (iii) For every $w' \in W$, $\mathcal{F}_{w'} B_w(\Lambda)$ is well-defined, and $\mathcal{F}_{w'} B_w(\Lambda) \cong B_{w''}(\Lambda)$ for some $w'' \in W$.

Proof. Since $\tilde{\tau} B(\Lambda) \cong B(\tau(\Lambda))$ and $\tilde{\tau}(\mathcal{F}_i S) = \mathcal{F}_{\tau(i)} \tilde{\tau}(S)$ for every $S \subseteq B(\Lambda)$, (i) follows. When $\ell(s_i w) = \ell(w) + 1$, (8) follows by definition, and when $\ell(s_i w) = \ell(w) - 1$, (8) follows since

$$\mathcal{F}_i B_w(\Lambda) = \mathcal{F}_i (\mathcal{F}_i B_{s_i w}(\Lambda)) = \mathcal{F}_i B_{s_i w}(\Lambda) = B_w(\Lambda).$$

Hence (ii) follows. To see that $\mathcal{F}_{w'} B_w(\Lambda)$ is well-defined, it suffices to show the operators \mathcal{F}_i on Demazure crystals satisfy braid relations: if the order of $s_i s_j$ for $i, j \in I$ ($i \neq j$) is $m < \infty$, then we have $\underbrace{\mathcal{F}_i \mathcal{F}_j \mathcal{F}_i \cdots}_m B_w(\Lambda) =$

$\underbrace{\mathcal{F}_j \mathcal{F}_i \mathcal{F}_j \cdots}_m B_w(\Lambda)$. Since the element $\underbrace{s_i s_j s_i \cdots}_m = \underbrace{s_j s_i s_j \cdots}_m$ is the longest element of $W_{i,j} = \langle s_i, s_j \rangle \subseteq W$, (ii) implies $\underbrace{\mathcal{F}_i \mathcal{F}_j \mathcal{F}_i \cdots}_m B_w(\Lambda) = B_{w''}(\Lambda) = \underbrace{\mathcal{F}_j \mathcal{F}_i \mathcal{F}_j \cdots}_m B_w(\Lambda)$ where w'' is the unique element of the set $\{\sigma w \mid \sigma \in W_{i,j}\}$ whose length is maximal. Hence the assertion follows. The second statement of (iii) is obvious from (ii). \square

For $w \in W$ and $\tau \in \text{Aut}(\Gamma)$, we write $\mathcal{F}_{w\tau} = \mathcal{F}_w \tilde{\tau}$ and $B_{w\tau}(\Lambda) = B_w(\tau(\Lambda))$ for the notational convenience. The following proposition is immediate from Lemma 4.3.

Proposition 4.4. *For every $\Lambda \in P^+$ and $w, w' \in \widetilde{W}$, there exists $w'' \in \widetilde{W}$ such that*

$$\mathcal{F}_{w'} B_w(\Lambda) \cong B_{w''}(\Lambda).$$

Let $\mathbb{C}[P]$ denote the group algebra of P with basis e^λ ($\lambda \in P$), and define for $i \in I$ a linear operator D_i on $\mathbb{C}[P]$ by

$$D_i(f) = \frac{f - e^{-\alpha_i} \cdot s_i(f)}{1 - e^{-\alpha_i}}$$

called the *Demazure operator*, where s_i acts on $\mathbb{C}[P]$ by $s_i(e^\lambda) = e^{s_i(\lambda)}$. Note that $D_i(f) = f$ if f is s_i -invariant. From this, it is easily checked that $D_i^2 = D_i$.

For every reduced expression $w = s_{i_k} \cdots s_{i_1}$ of $w \in W$, the operator $D_w = D_{i_k} \cdots D_{i_1}$ on $\mathbb{C}[P]$ is independent of the choice of the decomposition [17]. The weight sum of a Demazure crystal is known to be expressed using Demazure operators:

Theorem 4.5. [14] For $\Lambda \in P^+$ and $w \in W$, we have

$$\sum_{b \in B_w(\Lambda)} e^{\text{wt}(b)} = D_w(e^\Lambda).$$

For $w \in W$ and $\tau \in \Sigma$, we define an operator $D_{w\tau}$ on $\mathbb{C}[P]$ by $D_{w\tau} = D_w \circ \tau$, where τ acts on $\mathbb{C}[P]$ by $\tau(e^\Lambda) = e^{\tau(\Lambda)}$.

Corollary 4.6. Let S be a disjoint union of Demazure crystals and $i \in I$. For every $w \in \widehat{W}$ we have

$$\sum_{b \in \mathcal{F}_w(S)} e^{\text{wt}(b)} = D_w \left(\sum_{b \in S} e^{\text{wt}(b)} \right). \quad (9)$$

Proof. We may assume that S is a single Demazure crystal, say $S = B_{w'}(\Lambda)$. By Proposition 4.4, it suffices to show the assertion for $w = \tau \in \Sigma$ and $w = s_i$ for $i \in I$. When $w = \tau \in \Sigma$, the assertion is obvious from (7). Assume that $w = s_i$. If $\ell(s_i w') = \ell(w') + 1$, then $\mathcal{F}_i B_{w'}(\Lambda) = B_{s_i w'}(\Lambda)$ and the assertion follows from Theorem 4.5. If $\ell(s_i w') = \ell(w') - 1$, then we have $\mathcal{F}_i B_{w'}(\Lambda) = B_{w'}(\Lambda)$. On the other hand, we have

$$D_i \left(\sum_{b \in B_{w'}(\Lambda)} e^{\text{wt}(b)} \right) = \sum_{b \in B_{w'}(\Lambda)} e^{\text{wt}(b)}$$

since the weight sum

$$\sum_{b \in B_{w'}(\Lambda)} e^{\text{wt}(b)} = D_i \left(\sum_{b \in B_{s_i w'}(\Lambda)} e^{\text{wt}(b)} \right)$$

is s_i -invariant. Hence the assertion follows. \square

It is known that $B(\Lambda) \otimes B(\Lambda')$ for $\Lambda, \Lambda' \in P^+$ is isomorphic to a direct sum of the crystal bases of integrable highest weight modules, that is,

$$B(\Lambda) \otimes B(\Lambda') \cong \bigoplus_{\lambda \in T} B(\lambda), \quad (10)$$

where T is a possibly infinite multiset of elements of P^+ . The following theorem, which was proved in [20, Proposition 12] and [8, Theorem 2.11], is known as the combinatorial excellent filtration theorem:

Theorem 4.7. The image of the subset $u_\Lambda \otimes B_w(\Lambda')$ of $B(\Lambda) \otimes B(\Lambda')$ under the isomorphism $B(\Lambda) \otimes B(\Lambda') \cong \bigoplus_{\lambda \in T} B(\lambda)$ is a disjoint union of Demazure crystals.

5 Perfect Kirillov-Reshetikhin crystals

From this section to the end of the article, we assume that the type of \mathfrak{g} is nonexceptional (i.e. one of the types $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$). Note that some of the statements below on Kirillov-Reshetikhin crystals may have not been proved or not be true for exceptional \mathfrak{g} .

5.1 Kirillov-Reshetikhin crystals

For a $U'_q(\mathfrak{g})$ -crystal B , define two maps $\varepsilon, \varphi: B \rightarrow P_{\text{cl}}^+$ by

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i, \quad \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i \quad \text{for } b \in B.$$

Note that $\text{wt}(b) = \varphi(b) - \varepsilon(b)$.

Kirillov-Reshetikhin modules $W^{r,s}$ (KR modules, for short) are irreducible finite-dimensional $U'_q(\mathfrak{g})$ -modules parametrized by $r \in I_0$ and $s \in \mathbb{Z}_{\geq 1}$ (see [5] for the precise definition). For nonexceptional \mathfrak{g} , the following theorem is known:

Theorem 5.1 ([12, 24, 25]). *For each $r \in I_0$ and $s \in \mathbb{Z}_{\geq 1}$, the KR module $W^{r,s}$ has a crystal basis $B^{r,s}$.*

The crystals $B^{r,s}$ are called the *Kirillov-Reshetikhin crystals* (KR crystals, for short). Let \mathcal{C} denote the set consisting of tensor products of KR crystals.

Let B be a regular crystal. An element $b \in B$ is called *extremal* if for every $w \in W$ and $i \in I$,

$$e_i S_w(b) = 0 \text{ if } \langle \text{wt}(S_w(b)), \alpha_i^\vee \rangle \geq 0 \text{ and } f_i S_w(b) = 0 \text{ if } \langle \text{wt}(S_w(b)), \alpha_i^\vee \rangle \leq 0.$$

Definition 5.2 ([1]). A finite regular $U'_q(\mathfrak{g})$ -crystal B is called *simple* if there exists $\lambda \in P_{\text{cl}}^0$ such that B has a unique element whose weight is λ , the weights of B are contained in the convex hull of $W\lambda$, and the weight of each extremal element is in $W\lambda$.

Proposition 5.3 ([21, Proposition.3.8 (1)]). *Every $B \in \mathcal{C}$ is simple.*

Since $B \in \mathcal{C}$ is simple, B has a unique extremal element $u(B)$ such that $\langle \text{wt}(u(B)), \alpha_i^\vee \rangle \geq 0$ for all $i \in I_0$. It is known that $u(B^{r,s})$ is the unique element with weight $s\varpi_r$, and we have $u(B_1 \otimes B_2) = u(B_1) \otimes u(B_2)$ for $B_1, B_2 \in \mathcal{C}$. Every $B \in \mathcal{C}$ is connected by [1, Lemma 1.9 and 1.10]. Then by [10], we have the following:

Lemma 5.4 ([10, Lemma 3.3 (b)]). *For $B \in \mathcal{C}$ and every $b \in B$, we have*

$$B = \{e_{i_k} \cdots e_{i_1}(b) \mid k \geq 0, i_j \in I\} \setminus \{0\}.$$

The following proposition is important:

Proposition 5.5. *Let $B \in \mathcal{C}$. For every $\tau \in \Sigma$, there exists a unique isomorphism $\rho_\tau: \tilde{\tau}(B) \xrightarrow{\sim} B$ of $U'_q(\mathfrak{g})$ -crystals.*

Proof. For a single KR crystal $B = B^{r,s}$, $\tilde{\tau}(B^{r,s}) \cong B^{r,s}$ is proved in [28, Lemma 6.5], and this implies $\tilde{\tau}(B) \cong B$ for general $B \in \mathcal{C}$ since $\tilde{\tau}(B_1 \otimes B_2) \cong \tilde{\tau}(B_1) \otimes \tilde{\tau}(B_2)$. The uniqueness of the isomorphism follows since B is connected and an element of B with weight $\text{wt}(u(B))$ is unique. \square

We define an action of Σ on $B \in \mathcal{C}$ by $\tau(b) = \rho_\tau(\tilde{\tau}(b))$ for $\tau \in \Sigma$. This action satisfies

$$\tau \circ e_i = e_{\tau(i)} \circ \tau \text{ and } \tau \circ f_i = f_{\tau(i)} \circ \tau \quad \text{for all } i \in I, \quad (11)$$

where $\tau(0)$ is understood as 0.

Lemma 5.6. *For every $\tau \in \Sigma$, there exists some $w \in W_0$ such that*

$$\tau(u(B)) = S_w u(B) \quad \text{for all } B \in \mathcal{C}.$$

Proof. Since $\tau \in \widetilde{W} = W_0 \ltimes T(\widetilde{M})$ and $T(\widetilde{M})$ acts on P_{cl}^0 trivially, there exists $w \in W_0$ such that $\tau|_{P_{\text{cl}}^0} = w|_{P_{\text{cl}}^0}$. Then since

$$\text{wt}(\tau(u(B))) = \tau(\text{wt}(u(B))) = w(\text{wt}(u(B))) = \text{wt}(S_w u(B)),$$

$\tau(u(B)) = S_w u(B)$ follows by Proposition 5.3. \square

The $U_q(\mathfrak{g}_0)$ -crystal structure of a KR crystal is known by [2, 7]. In particular, we have the following proposition (for nonexceptional \mathfrak{g}):

Proposition 5.7. *A KR crystal $B^{r,s}$ is multiplicity free as a $U_q(\mathfrak{g}_0)$ -crystal. In other words, any two distinct $U_q(\mathfrak{g}_0)$ -components of $B^{r,s}$ are not isomorphic as $U_q(\mathfrak{g}_0)$ -crystals.*

Corollary 5.8. *Let $b_1, b_2 \in B^{r,s}$ be two distinct $U_q(\mathfrak{g}_0)$ -highest weight elements. Then we have*

$$\varphi(b_1) - \varphi(b_2) \notin \mathbb{Z}\Lambda_0.$$

Proof. For $j = 1, 2$, let $B_j \subseteq B^{r,s}$ be the $U_q(\mathfrak{g}_0)$ -component containing b_j . Then as $U_q(\mathfrak{g}_0)$ -crystals, B_j is isomorphic to the crystal basis of the integrable highest weight $U_q(\mathfrak{g}_0)$ -module with highest weight $\sum_{i \in I_0} \varphi_i(b_j) \varpi_i$. Now, the assertion is obvious from the above proposition. \square

5.2 Perfect KR crystals

For a $U'_q(\mathfrak{g})$ -crystal B such that $\text{wt}(B) \subseteq P_{\text{cl}}^0$, we define the *level* of B by

$$\text{lev}(B) = \min_{b \in B} \langle \varphi(b), K \rangle = \min_{b \in B} \langle \varepsilon(b), K \rangle,$$

and the subset B_{\min} by

$$B_{\min} = \{b \in B \mid \langle \varphi(b), K \rangle = \text{lev}(B)\} \tag{12}$$

$$= \{b \in B \mid \langle \varepsilon(b), K \rangle = \text{lev}(B)\}. \tag{13}$$

Definition 5.9 ([11]). For a positive integer ℓ , a $U'_q(\mathfrak{g})$ -crystal B is called a *perfect crystal of level ℓ* if B satisfies the following conditions:

- (i) B is isomorphic to the crystal basis of a finite-dimensional $U'_q(\mathfrak{g})$ -module.
- (ii) $B \otimes B$ is connected.
- (iii) There exists $\lambda \in P_{\text{cl}}^0$ such that $\text{wt}(B) \subseteq \lambda - \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$ and there exists a unique element in B with weight λ .
- (iv) The level of B is ℓ .
- (v) Both the maps ε and φ induce bijections between the set B_{\min} and $(P_{\text{cl}}^+)^{\ell}$.

The following lemma is immediate:

Lemma 5.10. *Let B_1, B_2 be perfect crystals.*

- (i) $\text{lev}(B_1 \otimes B_2) = \max\{\text{lev}(B_1), \text{lev}(B_2)\}$.
- (ii) *If $\text{lev}(B_1) \geq \text{lev}(B_2)$, then $b_1 \otimes b_2 \in B_1 \otimes B_2$ belongs to $(B_1 \otimes B_2)_{\min}$ if and only if $b_1 \in (B_1)_{\min}$ and $\varphi(b_1) - \varepsilon(b_2) \in P_{\text{cl}}^+$. Moreover if $b_1 \otimes b_2 \in (B_1 \otimes B_2)_{\min}$, then*

$$\varepsilon(b_1 \otimes b_2) = \varepsilon(b_1), \quad \varphi(b_1 \otimes b_2) = \varphi(b_1) + \text{wt}(b_2).$$

- (iii) *If $\text{lev}(B_1) \leq \text{lev}(B_2)$, then $b_1 \otimes b_2 \in B_1 \otimes B_2$ belongs to $(B_1 \otimes B_2)_{\min}$ if and only if $b_2 \in (B_2)_{\min}$ and $\varepsilon(b_2) - \varphi(b_1) \in P_{\text{cl}}^+$. Moreover if $b_1 \otimes b_2 \in (B_1 \otimes B_2)_{\min}$, then*

$$\varepsilon(b_1 \otimes b_2) = \varepsilon(b_2) - \text{wt}(b_1), \quad \varphi(b_1 \otimes b_2) = \varphi(b_2).$$

The significance of the perfectness is due to the following theorem:

Theorem 5.11 ([11]). *Let B be a perfect crystal of level ℓ , $\Lambda \in P^+$ a dominant integral weight of level ℓ , and b the unique element of B satisfying $\varepsilon(b) = \text{cl}(\Lambda)$. Then for all $\Lambda' \in P^+$ such that $\varphi(b) = \text{cl}(\Lambda')$, we have*

$$B(\Lambda) \otimes B \xrightarrow{\sim} B(\Lambda')$$

as $U'_q(\mathfrak{g})$ -crystals, and this isomorphism maps $u_\Lambda \otimes b$ to $u_{\Lambda'}$.

If B is a perfect crystal of level ℓ , then $\varepsilon \circ \varphi^{-1}$ induces a bijection $(P_{\text{cl}}^+)^{\ell} \rightarrow (P_{\text{cl}}^+)^{\ell}$, which is called the *associated automorphism* of B .

For $i \in I$, we denote by $\tau^i \in \Sigma$ the unique element satisfying $t_{c_i \varpi_i}(\tau^i)^{-1} \in W$. Note that this definition is the same as that of Subsection 2.2 for $i \in I^s$. For $i \in I \setminus I^s$, τ^i are as follows: for $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $\tau^i = \text{id}$ if i is even, and $\tau^i = \tau^1$ if i is odd. For $C_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, $\tau^i = \text{id}$ for all $i \in I \setminus I^s$.

Theorem 5.12 ([3]).

- (i) *The level of a KR crystal $B^{r,s}$ is $\lceil s/c_r \rceil (= \min\{m \in \mathbb{Z} \mid m \geq s/c_r\})$.*
- (ii) *$B^{r,s}$ is perfect if and only if $s/c_r \in \mathbb{Z}$.*
- (iii) *The associated automorphism of $B^{r,c_r s}$ coincides with the action of $(\tau^r)^{-1}$ on $(P_{\text{cl}}^+)^{\ell}$.*

Proof. The assertions (i) and (ii) are just proved in [3]. The associated automorphism of each $B^{r,c_r s}$ is explicitly described in [3], and we can check the assertion (iii) directly from them. We remark that the equation in [3, Subsection 4.3] for the associated automorphism of $B^{n,s}$ for $D_n^{(1)}$ is misprint. It should be modified as follows:

$$\tau\left(\sum_{i=0}^n \ell_i \Lambda_i\right) = \ell_n \Lambda_0 + \ell_{n-1} \Lambda_1 + \sum_{i=2}^{n-2} \ell_i \Lambda_{n-i} + \begin{cases} \ell_0 \Lambda_{n-1} + \ell_1 \Lambda_n & n \text{ even,} \\ \ell_1 \Lambda_{n-1} + \ell_0 \Lambda_n & n \text{ odd.} \end{cases}$$

□

Let $B = B^{r,s}$ be a (not necessarily perfect) KR crystal. B is known to have a unique element belonging to B_{\min} , which we denote by $m(B)$, such that

$$\varepsilon(m(B)) = \text{lev}(B) \Lambda_0.$$

(If B is perfect, this fact follows from the definition. For non-perfect ones, see [21, Lemma 3.11]). Similarly, B has a unique element $m'(B) \in B_{\min}$ such that

$$\varphi(m'(B)) = \text{lev}(B)\Lambda_0.$$

If B is perfect, we have from Theorem 5.12 (iii) that

$$\varphi(m(B)) = \text{lev}(B)\Lambda_{\tau^r(0)}.$$

The following theorem connects a perfect KR crystal with a Demazure crystal:

Theorem 5.13 ([28, Theorem 6.1]). *Let $B = B^{r, c_{rs}}$ be a perfect KR crystal. Then the isomorphism*

$$B(s\Lambda_0) \otimes B \xrightarrow{\sim} B(s\Lambda_{\tau^r(0)})$$

given in Theorem 5.11 maps the subset $u_{s\Lambda_0} \otimes B$ onto the Demazure crystal $B_{t_{c_r w_0(\varpi_r)}}(s\Lambda_0)$.

Later we need the following lemma:

Lemma 5.14. *Let B_1, B_2 be perfect KR crystals, and assume that $\text{lev}(B_1) \leq \text{lev}(B_2)$. If $b_1 \otimes b_2 \in (B_1 \otimes B_2)_{\min}$, then for every $b'_2 \in B_2$ there exists a sequence i_1, \dots, i_k of elements of I such that*

$$\begin{aligned} e_{i_k} \cdots e_{i_1}(b_1 \otimes b'_2) &= b_1 \otimes (e_{i_k} \cdots e_{i_1} b'_2) \\ &= b_1 \otimes b_2. \end{aligned} \tag{14}$$

Proof. By Lemma 5.10 (iii), $b_2 \in (B_2)_{\min}$ and $\varepsilon(b_2) - \varphi(b_1) \in P_{\text{cl}}^+$. Let $\Lambda = \text{aff}(\varepsilon(b_2))$ and $\Lambda' = \text{aff}(\varphi(b_2))$. Then by Theorem 5.11, there exists an isomorphism

$$B(\Lambda) \otimes B_2 \xrightarrow{\sim} B(\Lambda')$$

which maps $u_\Lambda \otimes b_2$ to $u_{\Lambda'}$. Therefore, there exists a sequence i_1, \dots, i_k of elements of I such that

$$\begin{aligned} e_{i_k} \cdots e_{i_1}(u_\Lambda \otimes b'_2) &= u_\Lambda \otimes (e_{i_k} \cdots e_{i_1} b'_2) \\ &= u_\Lambda \otimes b_2. \end{aligned}$$

The above equation implies that $\varepsilon_{i_q}(e_{i_{q-1}} \cdots e_{i_1} b'_2) > \varphi_{i_q}(u_\Lambda) = \varepsilon_{i_q}(b_2)$ for each $1 \leq q \leq k$. Then it follows for each $1 \leq q \leq k$ that

$$\varepsilon_{i_q}(e_{i_{q-1}} \cdots e_{i_1} b'_2) > \varepsilon_{i_q}(b_2) \geq \varphi_{i_q}(b_1),$$

and hence (14) holds. \square

5.3 Isomorphism as full subgraphs of $U'_q(\mathfrak{g})$ -crystals

We need the following elementary lemma:

Lemma 5.15. *Let B_1, B_2 be crystals, and $b_j \in B_j$ ($j = 1, 2$) arbitrary elements. If $f_i b_1 \neq 0$ for $i \in I$, then there exist some $b'_2 \in B_2$ and $m \in \mathbb{Z}_{>0}$ such that*

$$f_i(b_1) \otimes b_2 = f_i^m(b_1 \otimes b'_2).$$

Proof. When $\varphi_i(b_1) > \varepsilon_i(b_2)$, $m = 1$ and $b'_2 = b_2$ satisfy the assertion, and when $\varphi_i(b_1) \leq \varepsilon_i(b_2)$, $m = \varepsilon_i(b_2) - \varphi_i(b_1) + 2$ and $b'_2 = e_i^{m-1}b_2$ satisfy it. \square

Proposition 5.16. *Let $B_j = B^{r_j, c_{r_j} s_j}$ for $1 \leq j \leq p$ be perfect KR crystals, and assume that $s_1 \leq s_2 \leq \dots \leq s_p$. We put $\mu_j = c_{r_j} w_0(\varpi_{r_j})$ for $1 \leq j \leq p$, and $B = B_p \otimes \dots \otimes B_2 \otimes B_1$. Then there exists an isomorphism*

$$\Psi_B: u_{s_p \Lambda_0} \otimes B \xrightarrow{\sim} \mathcal{F}_{t_{\mu_p}} \left(u_{(s_p - s_{p-1}) \Lambda_0} \otimes \dots \otimes \mathcal{F}_{t_{\mu_2}} \left(u_{(s_2 - s_1) \Lambda_0} \otimes \mathcal{F}_{t_{\mu_1}}(u_{s_1 \Lambda_0}) \right) \dots \right)$$

of full subgraphs of $U'_q(\mathfrak{g})$ -crystals.

Proof. We show the assertion by the induction on p . If $p = 1$, then the assertion follows from Theorem 5.13. Assume $p > 1$. We put $\tau = \tau^{r_p}$ and $w = t_{c_{r_p} w_0(\varpi_{r_p})}(\tau^{r_p})^{-1} \in W$. Since $u_{s_p \Lambda_0} \otimes B_p \xrightarrow{\sim} \mathcal{F}_w(u_{s_p \Lambda_{\tau(0)}})$, we have

$$u_{s_p \Lambda_0} \otimes B_p \otimes \dots \otimes B_1 \xrightarrow{\sim} \mathcal{F}_w(u_{s_p \Lambda_{\tau(0)}}) \otimes B_{p-1} \otimes \dots \otimes B_1,$$

and we have from Lemma 5.15 that

$$\mathcal{F}_w(u_{s_p \Lambda_{\tau(0)}}) \otimes B_{p-1} \otimes \dots \otimes B_1 = \mathcal{F}_w(u_{s_p \Lambda_{\tau(0)}} \otimes B_{p-1} \otimes \dots \otimes B_1).$$

Since we have from Proposition 5.5 that

$$\begin{aligned} u_{s_p \Lambda_{\tau(0)}} \otimes B_{p-1} \otimes \dots \otimes B_1 &\cong \tilde{\tau}(u_{s_p \Lambda_0} \otimes B_{p-1} \otimes \dots \otimes B_1) \\ &\cong \tilde{\tau}(u_{(s_p - s_{p-1}) \Lambda_0} \otimes (u_{s_{p-1} \Lambda_0} \otimes B_{p-1} \otimes \dots \otimes B_1)), \end{aligned}$$

the induction hypothesis implies the assertion. \square

Remark 5.17. (i) Put $B^{p-1} = B_{p-1} \otimes \dots \otimes B_2 \otimes B_1$. We see from the construction of the isomorphism Ψ_B that the following diagram of set maps commutes (where we set $\tau = \tau^{r_p}$):

$$\begin{array}{ccc} u_{s_{p-1} \Lambda_0} \otimes B_{p-1} \otimes \dots \otimes B_1 & \xrightarrow[\Psi_{B^{p-1}}]{\cong} & \mathcal{F}_{t_{\mu_{p-1}}}(u_{(s_{p-1} - s_{p-2}) \Lambda_0} \otimes \dots) \\ \cong \downarrow \varphi & & \cong \downarrow \psi \\ u_{s_p \Lambda_0} \otimes m(B_p) \otimes B_{p-1} \otimes \dots \otimes B_1 & & \tilde{\tau}(u_{(s_p - s_{p-1}) \Lambda_0} \otimes \mathcal{F}_{t_{\mu_{p-1}}}(\dots)) \\ \downarrow & & \downarrow \\ u_{s_p \Lambda_0} \otimes B_p \otimes B_{p-1} \otimes \dots \otimes B_1 & \xrightarrow[\Psi_B]{\cong} & \mathcal{F}_{t_{\mu_p}}(u_{(s_p - s_{p-1}) \Lambda_0} \otimes \mathcal{F}_{t_{\mu_{p-1}}}(\dots)), \end{array}$$

where the isomorphisms φ and ψ are defined by

$$\varphi(u_{s_{p-1} \Lambda_0} \otimes b) = u_{s_p \Lambda_0} \otimes m(B_p) \otimes \tau(b) \quad \text{for } b \in B^{p-1}$$

and

$$\psi(b) = \tilde{\tau}(u_{(s_p - s_{p-1}) \Lambda_0} \otimes b) \quad \text{for } b \in \mathcal{F}_{t_{\mu_{p-1}}}(u_{(s_{p-1} - s_{p-2}) \Lambda_0} \otimes \dots)$$

respectively.

(ii) By Proposition 4.4 and Theorem 4.7, the right hand side of the isomorphism Ψ_B is isomorphic as full subgraphs to a disjoint union of Demazure crystals.

(iii) The right hand side of Ψ_B also appeared in [20] as the crystal basis of a generalized Demazure module.

Note that the right hand side of Ψ_B is a subset of a tensor product of the crystal bases of $U_q(\mathfrak{g})$ -modules. Hence each element b of this set has a natural \mathbb{Z} -grading defined by $\langle \text{wt}(b), d \rangle$. The goal of this article is to show, under the isomorphism Ψ_B , the minus of this natural grading coincides up to a shift with the grading on the left hand side given by the energy function, which is introduced in the next section.

6 Energy function

Similarly as [11], the following proposition is proved from the existence of the universal R -matrix and Theorem 5.1:

Proposition 6.1. *Let $B_1, B_2 \in \mathcal{C}$.*

- (i) *There exists a unique isomorphism $\sigma = \sigma_{B_1, B_2}: B_1 \otimes B_2 \xrightarrow{\sim} B_2 \otimes B_1$ of $U'_q(\mathfrak{g})$ -crystals called the combinatorial R -matrix.*
- (ii) *There exists a unique map $H = H_{B_1, B_2}: B_1 \otimes B_2 \rightarrow \mathbb{Z}$ called the local energy function such that $H(u(B_1 \otimes B_2)) = 0$, H is constant on each $U_q(\mathfrak{g}_0)$ -component, and for $b_1 \otimes b_2 \in B_1 \otimes B_2$ mapped to $\tilde{b}_2 \otimes \tilde{b}_1 \in B_2 \otimes B_1$ under σ , we have*

$$H(e_0(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) + 1 & \text{if } e_0(b_1 \otimes b_2) = e_0 b_1 \otimes b_2, \quad e_0(\tilde{b}_2 \otimes \tilde{b}_1) = e_0 \tilde{b}_2 \otimes \tilde{b}_1, \\ H(b_1 \otimes b_2) - 1 & \text{if } e_0(b_1 \otimes b_2) = b_1 \otimes e_0 b_2, \quad e_0(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{b}_2 \otimes e_0 \tilde{b}_1, \\ H(b_1 \otimes b_2) & \text{otherwise.} \end{cases}$$

For $B_1, B_2 \in \mathcal{C}$, we have $\sigma(u(B_1) \otimes u(B_2)) = u(B_2) \otimes u(B_1)$ by the weight consideration. Recall that for every $\tau \in \Sigma$, there exists some $w \in W_0$ such that $\tau(u(B_1) \otimes u(B_2)) = S_w(u(B_1) \otimes u(B_2))$ by Lemma 5.6. Hence we have

$$\begin{aligned} \sigma \circ \tau(u(B_1) \otimes u(B_2)) &= \sigma \circ S_w(u(B_1) \otimes u(B_2)) \\ &= S_w(u(B_2) \otimes u(B_1)) = \tau(u(B_2) \otimes u(B_1)), \end{aligned}$$

which together with (11) implies that σ commutes with the action of τ .

The following lemma is a consequence of the definition of the local energy function:

Lemma 6.2. *Let $B_1, B_2 \in \mathcal{C}$, $b_j \in B_j$ for $j = 1, 2$ such that $\sigma(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$, and j_1, \dots, j_ℓ an arbitrary sequence of elements of I satisfying $e_{j_\ell} \cdots e_{j_1}(b_1 \otimes b_2) \neq 0$. If*

$$\begin{aligned} e_{j_\ell} \cdots e_{j_1}(b_1 \otimes b_2) &= e_{i'_{\ell-k}} \cdots e_{i'_1} b_1 \otimes e_{i_k} \cdots e_{i_1} b_2 \quad \text{and} \\ e_{j_\ell} \cdots e_{j_1}(\tilde{b}_2 \otimes \tilde{b}_1) &= e_{\tilde{i}_m} \cdots e_{\tilde{i}_1} \tilde{b}_2 \otimes e_{\tilde{i}'_{\ell-m}} \cdots e_{\tilde{i}'_1} \tilde{b}_1 \end{aligned}$$

where

$$\{j_1, \dots, j_\ell\} = \{i_1, \dots, i_k\} \sqcup \{i'_1, \dots, i'_{\ell-k}\} = \{\tilde{i}_1, \dots, \tilde{i}_m\} \sqcup \{\tilde{i}'_1, \dots, \tilde{i}'_{\ell-m}\}$$

as multisets, then we have

$$\begin{aligned} H(e_{j_\ell} \cdots e_{j_1}(b_1 \otimes b_2)) - H(b_1 \otimes b_2) \\ = \#\{1 \leq q \leq m \mid \tilde{i}_q = 0\} - \#\{1 \leq q \leq k \mid i_q = 0\}. \end{aligned}$$

For $B \in \mathcal{C}$, the *energy function* $D = D_B: B \rightarrow \mathbb{Z}$ is defined as follows:
(i) If B is a single KR crystal, then define

$$D_B(b) = H_{B,B}(m'(B) \otimes b) - H_{B,B}(m'(B) \otimes u(B)).$$

(ii) If $B_1, B_2 \in \mathcal{C}$ and $B = B_1 \otimes B_2$, then define

$$D_B(b_1 \otimes b_2) = D_{B_1}(b_1) + D_{B_2}(\tilde{b}_2) + H_{B_1, B_2}(b_1 \otimes b_2),$$

where $\sigma_{B_1, B_2}(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$.

Note that D_B is constant on each $U_q(\mathfrak{g}_0)$ -component of B by definition.

Proposition 6.3 ([27]). (i) For $B_1, B_2, B_3 \in \mathcal{C}$, we have

$$D_{(B_1 \otimes B_2) \otimes B_3} = D_{B_1 \otimes (B_2 \otimes B_3)}.$$

Hence for every $B \in \mathcal{C}$, the function D_B is well-defined.

(ii) Let $B = B_1 \otimes \cdots \otimes B_p \in \mathcal{C}$. For $b_1 \otimes \cdots \otimes b_p \in B$ and $1 \leq i \leq j \leq p$, define $b_j^{(i)} \in B_j$ by

$$\begin{aligned} B_i \otimes \cdots \otimes B_{j-1} \otimes B_j &\xrightarrow{\sim} B_j \otimes B_i \otimes \cdots \otimes B_{j-1} \\ b_i \otimes \cdots \otimes b_{j-1} \otimes b_j &\mapsto b_j^{(i)} \otimes \tilde{b}_i \otimes \cdots \otimes \tilde{b}_{j-1}. \end{aligned}$$

Then we have

$$D_B(b_1 \otimes \cdots \otimes b_p) = \sum_{1 \leq j \leq p} D_{B_j}(b_j^{(1)}) + \sum_{1 \leq j < k \leq p} H_{B_j, B_k}(b_j \otimes b_k^{(j+1)}).$$

Lemma 6.4. Let $B \in \mathcal{C}$ and $\ell = \text{lev}(B)$. If $b \in B$ satisfies $\varepsilon_0(b) > \ell$, then we have

$$D(e_0 b) = D(b) - 1.$$

Proof. We show the assertion by the induction on the number p of tensor factors of B . The case $p = 1$ follows since

$$D(e_0 b) = H(m'(B) \otimes e_0 b) = H(m'(B) \otimes b) - 1 = D(b) - 1.$$

Assume $p > 1$, and write $B = B_1 \otimes B_2$, $b = b_1 \otimes b_2$ with $b_j \in B_j$. Note that we have $\text{lev}(B_1) \leq \ell$ and $\text{lev}(B_2) \leq \ell$. Let $\tilde{b}_2 \otimes \tilde{b}_1 = \sigma(b_1 \otimes b_2)$. Then we can show the assertion by computing case by case. For example, assume $e_0(b_1 \otimes b_2) = e_0 b_1 \otimes b_2$ and $e_0(\tilde{b}_2 \otimes \tilde{b}_1) = e_0 \tilde{b}_2 \otimes \tilde{b}_1$. Then we have $\varepsilon_0(b_1) = \varepsilon_0(b_1 \otimes b_2) > \ell$ and $\varepsilon_0(\tilde{b}_2) = \varepsilon_0(\tilde{b}_2 \otimes \tilde{b}_1) > \ell$, which imply from the induction hypothesis that

$$\begin{aligned} D(e_0 b) &= D(e_0 b_1) + D(e_0 \tilde{b}_2) + H(e_0(b_1 \otimes b_2)) \\ &= (D(b_1) - 1) + (D(\tilde{b}_2) - 1) + (H(b_1 \otimes b_2) + 1) \\ &= D(b) - 1. \end{aligned}$$

The other cases are proved similarly. □

7 Main theorem

7.1 Statement and corollaries

Now, we state the main theorem of this article. This theorem is a generalization of [28, Theorem 7.4], in which $s_1 = s_2 = \dots = s_p$ is assumed.

Theorem 7.1. *Let $B_j = B^{r_j, c_{r_j} s_j}$ for $1 \leq j \leq p$ be perfect KR crystals, and assume that $s_1 \leq s_2 \leq \dots \leq s_p$. We put $\mu_j = c_{r_j} w_0(\varpi_{r_j})$ for $1 \leq j \leq p$, and $B = B_p \otimes \dots \otimes B_2 \otimes B_1$. Then there exists an isomorphism*

$$\Psi_B: u_{s_p \Lambda_0} \otimes B \xrightarrow{\sim} \mathcal{F}_{t_{\mu_p}} \left(u_{(s_p - s_{p-1}) \Lambda_0} \otimes \dots \otimes \mathcal{F}_{t_{\mu_2}} \left(u_{(s_2 - s_1) \Lambda_0} \otimes \mathcal{F}_{t_{\mu_1}} (u_{s_1 \Lambda_0}) \right) \dots \right)$$

of full subgraphs of $U_q(\mathfrak{g})$ -crystals satisfying

$$D(b) = -\langle \text{wt } \Psi_B(u_{s_p \Lambda_0} \otimes b), d \rangle + C \quad (15)$$

for every $b \in B$, where $C \in \mathbb{Z}$ is some global constant.

Recall that, as stated in Remark 5.17 (ii), the right hand side of Ψ_B is isomorphic as full subgraphs to a disjoint union of Demazure crystals. Hence we can see inductively using Corollary 4.6 that the following equation holds:

Corollary 7.2. *Under the notation and the assumptions of Theorem 7.1, we have*

$$\begin{aligned} e^{s_p \Lambda_0 + C\delta} \sum_{b \in B} e^{\text{aff wt}(b) - \delta D(b)} \\ = D_{t_{\mu_p}} \left(e^{(s_p - s_{p-1}) \Lambda_0} \dots D_{t_{\mu_2}} \left(e^{(s_2 - s_1) \Lambda_0} \cdot D_{t_{\mu_1}} (e^{s_1 \Lambda_0}) \right) \dots \right). \end{aligned}$$

Let $P_0^+ \subseteq P_{\text{cl}}^0$ denote the set of dominant integral weights of \mathfrak{g}_0 . (Recall that we identify the weight lattice of \mathfrak{g}_0 with P_{cl}^0 .) As [6, 5], the *one-dimensional configuration sum* $X(B, \mu, q) \in \mathbb{Z}[q, q^{-1}]$ for $\mu \in P_0^+$ is defined by

$$X(B, \mu, q) = \sum_{\substack{b \in B \\ e_i b = 0 \ (i \in I_0) \\ \text{wt}(b) = \mu}} q^{D(b)}.$$

Let $\text{ch } V_{\mathfrak{g}_0}(\mu)$ denote the character of the irreducible \mathfrak{g}_0 -module with highest weight μ . Since

$$\sum_{b \in B} q^{D(b)} e^{\text{wt}(b)} = \sum_{\mu \in P_0^+} X(B, \mu, q) \text{ch } V_{\mathfrak{g}_0}(\mu)$$

holds, we have the following corollary:

Corollary 7.3. *Under the notation and the assumptions of Theorem 7.1, we have*

$$\begin{aligned} q^{-C} \sum_{\mu \in P_{\text{cl}}^0} X(B, \mu, q) \text{ch } V_{\mathfrak{g}_0}(\mu) \\ = e^{-s_p \Lambda_0} D_{t_{\mu_p}} \left(e^{(s_p - s_{p-1}) \Lambda_0} \dots D_{t_{\mu_2}} \left(e^{(s_2 - s_1) \Lambda_0} \cdot D_{t_{\mu_1}} (e^{s_1 \Lambda_0}) \right) \dots \right), \end{aligned}$$

where we set $q = e^{-\delta}$ and consider $\text{ch } V_{\mathfrak{g}_0}(\mu)$ as an element of $\mathbb{C}[P]$ via the map $\text{aff}: P_{\text{cl}} \rightarrow P$.

Remark 7.4. Let η be a permutation of the set $\{1, \dots, p\}$, and put $B_\eta = B_{\eta(1)} \otimes \dots \otimes B_{\eta(p)}$. Then we have from [27, Lemma 2.15] that

$$D_{B_\eta}(\sigma_\eta(b)) = D_B(b) \quad \text{for every } b \in B,$$

where $\sigma_\eta: B \xrightarrow{\sim} B_\eta$ is the unique isomorphism. In particular, we have

$$X(B, \mu, q) = X(B_\eta, \mu, q).$$

Hence, the above theorem and corollaries with replaced B by B_η (and the right hand sides unchanged) also hold for every η .

7.2 Proof of the main theorem

In order to show the main theorem, it remains to show that the isomorphism Ψ_B constructed in Proposition 5.16 satisfies (15).

To show this, we prepare several lemmas.

Lemma 7.5. *Let $B_1, B_2 \in \mathcal{C}$ and $\tau \in \Sigma$. For $b_1 \otimes b_2 \in B_1 \otimes B_2$ mapped to $\tilde{b}_2 \otimes \tilde{b}_1 \in B_2 \otimes B_1$ under σ , we have*

$$H(b_1 \otimes b_2) - H(\tau(b_1 \otimes b_2)) = \langle \text{wt}(b_2) - \text{wt}(\tilde{b}_2), \varpi_{\tau^{-1}(0)}^\vee \rangle. \quad (16)$$

Proof. Although the proof is carried out in an almost same way as that of [21, Lemma 8.2], we give it for the reader's convenience.

The case $\tau = \text{id}$ is trivial. Hence we assume otherwise, and put $t = \tau^{-1}(0) \in I^s \setminus \{0\}$. If $b_1 = u(B_1)$ and $b_2 = u(B_2)$, we have from Lemma 5.6 that

$$H(\tau(u(B_1) \otimes u(B_2))) = H(u(B_1) \otimes u(B_2)) = 0,$$

and hence the left hand side of (16) is 0. On the other hand, since $\sigma(u(B_1) \otimes u(B_2)) = u(B_2) \otimes u(B_1)$ the right hand side is also 0, and the assertion follows in this case. Therefore by Lemma 5.4, it suffices to show for each $i \in I$ that if (16) holds and $e_i(b_1 \otimes b_2) \neq 0$, then (16) with replaced $b_1 \otimes b_2$ by $e_i(b_1 \otimes b_2)$ also holds. If $i \neq 0, t$, it is easy to see that the both sides of (16) do not change when $b_1 \otimes b_2$ is replaced by $e_i(b_1 \otimes b_2)$. Assume that $i = 0$. Since $t \neq 0$, we have

$$\begin{aligned} & \left(H(e_0(b_1 \otimes b_2)) - H(\tau \circ e_0(b_1 \otimes b_2)) \right) - \left(H(b_1 \otimes b_2) - H(\tau(b_1 \otimes b_2)) \right) \\ &= \left(H(e_0(b_1 \otimes b_2)) - H(b_1 \otimes b_2) \right) - \left(H(e_t \circ \tau(b_1 \otimes b_2)) - H(\tau(b_1 \otimes b_2)) \right) \\ &= \begin{cases} 1 & \text{if } e_0(b_1 \otimes b_2) = e_0 b_1 \otimes b_2, \quad e_0(\tilde{b}_2 \otimes \tilde{b}_1) = e_0 \tilde{b}_2 \otimes \tilde{b}_1, \\ -1 & \text{if } e_0(b_1 \otimes b_2) = b_1 \otimes e_0 b_2, \quad e_0(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{b}_2 \otimes e_0 \tilde{b}_1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (17)$$

On the other hand, putting $e_0(b_1 \otimes b_2) = b'_1 \otimes b'_2$ and $e_0(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{b}'_2 \otimes \tilde{b}'_1$, we easily check from (2) that

$$\langle \text{wt}(b'_2) - \text{wt}(\tilde{b}'_2), \varpi_t^\vee \rangle - \langle \text{wt}(b_2) - \text{wt}(\tilde{b}_2), \varpi_t^\vee \rangle$$

is equal to (17), which implies the assertion for $i = 0$. The case $i = t$ is similar. \square

For $B \in \mathcal{C}$ and $s \in \mathbb{Z}_{>0}$, we define a subset $\text{hw}_{\tilde{I}_0}^{\leq s}(B) \subseteq B$ by

$$\text{hw}_{\tilde{I}_0}^{\leq s}(B) = \{b \in B \mid b \text{ is } U_q(\mathfrak{g}_0)\text{-highest weight, } \varepsilon_0(b) \leq s\}.$$

Lemma 7.6. *Let $B_j = B^{r_j, c_{r_j} s_j}$ ($j = 1, 2$) be two perfect KR crystals, and assume $s_1 \geq s_2$. For every $b_2 \in \text{hw}_{\tilde{I}_0}^{\leq s_1}(B_2)$, we have*

$$\sigma_{B_1, B_2}(m(B_1) \otimes \tau^{r_1}(b_2)) = b_2 \otimes b_1$$

for some $b_1 \in B_1$.

Proof. Put $\tau = \tau^{r_1}$. Since

$$\varphi(m(B_1)) = s_1 \Lambda_{\tau(0)} \text{ and } \varepsilon(\tau(b_2)) = \varepsilon_0(b_2) \Lambda_{\tau(0)},$$

we have from Lemma 5.10 (ii) that $m(B_1) \otimes \tau(b_2) \in (B_1 \otimes B_2)_{\min}$, $\varepsilon(m(B_1) \otimes \tau(b_2)) = s_1 \Lambda_0$, and

$$\varphi(m(B_1) \otimes \tau(b_2)) = (s_1 - \varepsilon_0(b_2)) \Lambda_{\tau(0)} + \tau(\varphi(b_2)).$$

Put $\Lambda = (s_1 - \varepsilon_0(b_2)) \Lambda_{\tau(0)} + \tau(\varphi(b_2))$, and let $b'_2 \otimes b'_1 = \sigma(m(B_1) \otimes \tau(b_2))$. Since $b'_2 \otimes b'_1 \in (B_2 \otimes B_1)_{\min}$, we have from Lemma 5.10 (iii) that $b'_1 \in (B_1)_{\min}$ and $\varphi(b'_1) = \varphi(b'_2 \otimes b'_1) = \Lambda$. Hence from Theorem 5.12 (iii), we have

$$\varepsilon(b'_1) = \tau^{-1}(\Lambda) = (s_1 - \varepsilon_0(b_2)) \Lambda_0 + \varphi(b_2). \quad (18)$$

By Lemma 5.10 (iii), it follows that

$$\varphi(b'_2) = \text{wt}(b'_2) + \varepsilon(b'_2) = \varepsilon(b'_1) + \varepsilon(b'_2) - \varepsilon(b'_1 \otimes b'_2).$$

Then since b'_2 is $U_q(\mathfrak{g}_0)$ -highest weight and $\varepsilon(b'_2 \otimes b'_1) = \varepsilon(m(B_1) \otimes \tau(b_2)) = s_1 \Lambda_0$, we have from (18) that

$$\varphi(b'_2) \in \varphi(b_2) + \mathbb{Z} \Lambda_0,$$

which implies $b_2 = b'_2$ by Corollary 5.8. \square

Lemma 7.7. *Let $B_j = B^{r_j, c_{r_j} s_j}$ ($j = 1, 2$) be two perfect KR crystals, and assume $s_1 \geq s_2$. Then there exists some global constant C such that*

$$H(m(B_1) \otimes \tau(b_2)) = -\langle \text{wt}(b_2), \varpi_{\tau^{-1}(0)}^\vee \rangle + C$$

for every $b_2 \in \text{hw}_{\tilde{I}_0}^{\leq s_1}(B_2)$, where we put $\tau = \tau^{r_1}$.

Proof. Although the proof of this lemma is basically the same as that of [26, Lemma 4.7], we include it for the reader's convenience.

It suffices to show for $b_2, b_2^\dagger \in \text{hw}_{\tilde{I}_0}^{\leq s_1}(B_2)$ that

$$H(m(B_1) \otimes \tau(b_2^\dagger)) - H(m(B_1) \otimes \tau(b_2)) = -\langle \text{wt}(b_2^\dagger) - \text{wt}(b_2), \varpi_{\tau^{-1}(0)}^\vee \rangle.$$

By Lemma 7.6, we have

$$\sigma(m(B_1) \otimes \tau(b_2)) = b_2 \otimes b_1, \quad \sigma(m(B_1) \otimes \tau(b_2^\dagger)) = b_2^\dagger \otimes b_1^\dagger$$

for some $b_1, b_1^\dagger \in B_1$. By Lemma 5.4, there exists a sequence i_1, \dots, i_k of elements of I such that

$$e_{i_k} \cdots e_{i_1} b_2 = b_2^\dagger.$$

We choose such a sequence so that k is minimal. Then there exists a sequence $j_1, \dots, j_\ell \in I$ such that

$$\begin{aligned} e_{j_\ell} \cdots e_{j_1} (b_2 \otimes b_1) &= e_{i_k} \cdots e_{i_1} b_2 \otimes e_{i'_{\ell-k}} \cdots e_{i'_1} b_1 \\ &= b_2^\dagger \otimes e_{i'_{\ell-k}} \cdots e_{i'_1} b_1. \end{aligned}$$

Since $b_2^\dagger \otimes b_1^\dagger \in (B_2 \otimes B_1)_{\min}$, by Lemma 5.14 we may assume $e_{i'_{\ell-k}} \cdots e_{i'_1} b_1 = b_1^\dagger$. Then we have

$$e_{j_\ell} \cdots e_{j_1} (m(B_1) \otimes \tau(b_2)) = m(B_1) \otimes \tau(b_2^\dagger).$$

The two sequences $\tilde{i}_1, \dots, \tilde{i}_m$ and $\tilde{i}'_1, \dots, \tilde{i}'_{\ell-m}$ of elements of I are defined by

$$\begin{aligned} e_{j_\ell} \cdots e_{j_1} (m(B_1) \otimes \tau(b_2)) &= e_{\tilde{i}'_{\ell-m}} \cdots e_{\tilde{i}'_1} m(B_1) \otimes e_{\tilde{i}_m} \cdots e_{\tilde{i}_1} \tau(b_2) \\ &= m(B_1) \otimes \tau(b_2^\dagger) \end{aligned}$$

Since

$$e_{\tilde{i}_m} \cdots e_{\tilde{i}_1} \tau(b_2) = \tau(e_{\tau^{-1}(\tilde{i}_m)} \cdots e_{\tau^{-1}(\tilde{i}_1)} b_2) = \tau(b_2^\dagger),$$

we have $e_{\tau^{-1}(\tilde{i}_m)} \cdots e_{\tau^{-1}(\tilde{i}_1)} b_2 = b_2^\dagger$, which implies

$$\sum_{1 \leq q \leq m} \alpha_{\tau^{-1}(\tilde{i}_q)} - \sum_{1 \leq q \leq k} \alpha_{i_q} \in \mathbb{Z}_{\geq 0} \delta \quad (19)$$

by the minimality of k .

By repeating the above procedure interchanging the roles of b_2 and b_2^\dagger , we obtain sequences of elements of I satisfying the following:

$$\begin{aligned} e_{j_\ell^*} \cdots e_{j_1^*} (b_2^\dagger \otimes b_1^\dagger) &= e_{i_{k^*}^*} \cdots e_{i_1^*} b_2^\dagger \otimes e_{i'_{\ell-k^*}^*} \cdots e_{i'_{1^*}^*} b_1^\dagger \\ &= b_2^\dagger \otimes b_1^\dagger, \\ e_{j_\ell^*} \cdots e_{j_1^*} (m(B_1) \otimes \tau(b_2^\dagger)) &= e_{\tilde{i}'_{\ell-m^*}^*} \cdots e_{\tilde{i}'_{1^*}^*} m(B_1) \otimes e_{\tilde{i}_{m^*}^*} \cdots e_{\tilde{i}_{1^*}^*} \tau(b_2^\dagger) \\ &= m(B_1) \otimes \tau(b_2), \\ \sum_{1 \leq q \leq m^*} \alpha_{\tau^{-1}(\tilde{i}_q^*)} - \sum_{1 \leq q \leq k^*} \alpha_{i_q^*} &\in \mathbb{Z}_{\geq 0} \delta. \end{aligned} \quad (20)$$

By Lemma 6.2, we have

$$\begin{aligned} 0 &= \left(H(m(B_1) \otimes \tau(b_2^\dagger)) - H(m(B_1) \otimes \tau(b_2)) \right) \\ &\quad + \left(H(m(B_1) \otimes \tau(b_2)) - H(m(B_1) \otimes \tau(b_2^\dagger)) \right) \\ &= \left(\#\{1 \leq q \leq k \mid i_q = 0\} - \#\{1 \leq q \leq m \mid \tilde{i}_q = 0\} \right) \\ &\quad + \left(\#\{1 \leq q \leq k^* \mid i_q^* = 0\} - \#\{1 \leq q \leq m^* \mid \tilde{i}_q^* = 0\} \right) \\ &= \left(\#\{1 \leq q \leq k \mid i_q = 0\} + \#\{1 \leq q \leq k^* \mid i_q^* = 0\} \right) \\ &\quad - \left(\#\{1 \leq q \leq m \mid \tilde{i}_q = 0\} + \#\{1 \leq q \leq m^* \mid \tilde{i}_q^* = 0\} \right). \end{aligned}$$

Since $a_0 = a_{\tau^{-1}(0)}$ by (4), this equation together with (19) and (20) implies that $k = m$, $k^* = m^*$,

$$\begin{aligned}\{\tau^{-1}(\tilde{i}_1), \dots, \tau^{-1}(\tilde{i}_m)\} &= \{i_1, \dots, i_k\} \text{ and} \\ \{\tau^{-1}(\tilde{i}_1^*), \dots, \tau^{-1}(\tilde{i}_{m^*}^*)\} &= \{i_1^*, \dots, i_{k^*}^*\}\end{aligned}$$

as multisets. Hence we have

$$\begin{aligned}& H(m(B_1) \otimes \tau(b_2^\dagger)) - H(m(B_1) \otimes \tau(b_2)) \\ &= \#\{1 \leq q \leq k \mid i_q = 0\} - \#\{1 \leq q \leq m \mid \tilde{i}_q = 0\} \\ &= \#\{1 \leq q \leq k \mid i_q = 0\} - \#\{1 \leq q \leq k \mid i_q = \tau^{-1}(0)\} \\ &= -\langle \text{wt}(b_2^\dagger) - \text{wt}(b_2), \varpi_{\tau^{-1}(0)}^\vee \rangle,\end{aligned}$$

and the assertion is proved. \square

The following lemma is crucial for the proof of our theorem:

Lemma 7.8. *Let $B_j = B^{r_j, c_{r_j} s_j}(0 \leq j \leq p)$ be perfect KR crystals, and put $B = B_1 \otimes B_2 \otimes \dots \otimes B_p$. We assume that $s_0 \geq s_j$ for every $1 \leq j \leq p$. Then there exists some global constant C such that*

$$D(m(B_0) \otimes \tau(b)) = D(b) - \langle \text{wt}(b), \varpi_{\tau^{-1}(0)}^\vee \rangle + C$$

for every $b \in \text{hw}_{I_0}^{\leq s_0}(B)$, where we put $\tau = \tau^{r_0}$.

Proof. Let $b = b_1 \otimes \dots \otimes b_p \in \text{hw}_{I_0}^{\leq s_0}(B)$, and define $b_j^{(i)} \in B_j$ for $1 \leq i \leq j \leq p$ as Proposition 6.3 (ii). Note that, since the combinatorial R -matrix and the action of τ commute, the first tensor factor of the image of $\tau(b_i \otimes \dots \otimes b_j)$ under the isomorphism

$$B_i \otimes \dots \otimes B_{j-1} \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes \dots \otimes B_{j-1}$$

is $\tau(b_j^{(i)})$. Since $b \in \text{hw}_{I_0}^{\leq s_0}(B)$ implies $b_j^{(1)} \in \text{hw}_{I_0}^{\leq s_0}(B_j)$ for each $1 \leq j \leq p$, we have for each $1 \leq j \leq p$ that

$$\sigma_{B_0, B_j}(m(B_0) \otimes \tau(b_j^{(1)})) = b_j^{(1)} \otimes b^j$$

for some $b^j \in B_0$ by Lemma 7.6. Hence by Proposition 6.3 (ii), we have

$$\begin{aligned}D(m(B_0) \otimes \tau(b)) &= D(m(B_0)) + \sum_{1 \leq j \leq p} D(b_j^{(1)}) + \sum_{1 \leq j \leq p} H(m(B_0) \otimes \tau(b_j^{(1)})) \\ &\quad + \sum_{1 \leq j < k \leq p} H(\tau(b_j) \otimes \tau(b_k^{(j+1)})).\end{aligned}\tag{21}$$

For each $1 \leq j \leq p$ we have by Lemma 7.7 that

$$H(m(B_0) \otimes \tau(b_j^{(1)})) = -\langle \text{wt}(b_j^{(1)}), \varpi_{\tau^{-1}(0)}^\vee \rangle + C_j$$

for some constant C_j independent of $b_j^{(1)}$, and for each $1 \leq j < k \leq p$ we have by Lemma 7.5 that

$$H(\tau(b_j) \otimes \tau(b_k^{(j+1)})) = H(b_j \otimes b_k^{(j+1)}) - \langle \text{wt}(b_k^{(j+1)}) - \text{wt}(b_k^{(j)}), \varpi_{\tau^{-1}(0)}^\vee \rangle.$$

Hence, we have with some global constant C that

$$\begin{aligned}
(21) &= \sum_{1 \leq j \leq p} D(b_j^{(1)}) + \sum_{1 \leq j < k \leq p} H(b_j \otimes b_k^{(j+1)}) - \sum_{1 \leq j \leq p} \langle \text{wt}(b_j^{(1)}), \varpi_{\tau^{-1}(0)}^\vee \rangle \\
&\quad - \sum_{1 \leq j < k \leq p} \langle \text{wt}(b_k^{(j+1)}) - \text{wt}(b_k^{(j)}), \varpi_{\tau^{-1}(0)}^\vee \rangle + C \\
&= D(b) - \sum_{1 \leq k \leq p} \langle \text{wt}(b_k), \varpi_{\tau^{-1}(0)}^\vee \rangle + C \\
&= D(b) - \langle \text{wt}(b), \varpi_{\tau^{-1}(0)}^\vee \rangle + C.
\end{aligned}$$

Hence, the assertion follows. \square

Now, we give the proof of our main theorem:

Proof of Theorem 7.1. It remains to verify that Ψ_B in Proposition 5.16 satisfies (15) for every $b \in B$. First, we show the following claim.

claim. If every element $b \in \text{hw}_{I_0}^{\leq s_p}(B)$ satisfies (15), then all the elements of B satisfy (15).

Let $b \in B$ be an arbitrary element. Since $u_{s_p \Lambda_0} \otimes B$ is isomorphic to a disjoint union of Demazure crystals (see Remark 5.17 (ii)), there exists a sequence i_1, \dots, i_k of elements of I such that the element $e_{i_k} \cdots e_{i_1}(u_{s_p \Lambda_0} \otimes b) \neq 0$ is $U_q(\mathfrak{g})$ -highest weight. We show the claim by the induction on k . If $k = 0$, then $u_{s_p \Lambda_0} \otimes b$ itself is $U_q(\mathfrak{g})$ -highest weight, which is equivalent to $b \in \text{hw}_{I_0}^{\leq s_p}(B)$. Hence there is nothing to prove. Assume $k > 0$, and set $b' = e_{i_1}(b)$. Then (15) with replaced b by b' holds by the induction hypothesis. Note that if $i_1 = 0$, then $\varepsilon_0(b) > s_p$ since $e_0(u_{s_p \Lambda_0} \otimes b) = u_{s_p \Lambda_0} \otimes e_0 b$. Hence we have from Lemma 6.4 that

$$\begin{aligned}
D(b) &= D(b') + \delta_{0i_1} = -\langle \text{wt } \Psi_B(u_{s_p \Lambda_0} \otimes b'), d \rangle + \delta_{0i_1} + C \\
&= -\langle \text{wt } e_{i_1} \Psi_B(u_{s_p \Lambda_0} \otimes b), d \rangle + \delta_{0i_1} + C \\
&= -\langle \text{wt } \Psi_B(u_{s_p \Lambda_0} \otimes b), d \rangle + C,
\end{aligned}$$

and the claim is proved.

In particular, since the set $\text{hw}_{I_0}^{\leq s_1}(B_1)$ contains only a single element $m(B_1)$, the assertion of the theorem for $p = 1$ follows from the claim. Assume $p > 1$, and we show the assertion of the theorem by the induction on p . Put $B^{p-1} = B_{p-1} \otimes \cdots \otimes B_1$ and $\tau = \tau^{r_p}$. Let $b \in \text{hw}_{I_0}^{\leq s_p}(B)$ be an arbitrary element, and write $b = b_p \otimes b^{p-1} \in B_p \otimes B^{p-1}$. Since $\text{lev}(B_p) \geq \text{lev}(B^{p-1})$, $b \in \text{hw}_{I_0}^{\leq s_p}(B)$ implies by Lemma 5.10 (ii) that

$$b_p \in \text{hw}_{I_0}^{\leq s_p}(B_p) \quad \text{and} \quad \varepsilon(b^{p-1}) \in \varphi(b_p) - P_{\text{cl}}^+.$$

Then since $\text{lev}(B_p) = s_p$, these are equivalent to

$$b_p = m(B_p) \quad \text{and} \quad \varepsilon(b^{p-1}) \in s_p \Lambda_{\tau(0)} - P_{\text{cl}}^+.$$

Hence if we put $b' = \tau^{-1}(b^{p-1})$, we have

$$b = m(B_p) \otimes \tau(b') \quad \text{with } b' \in \text{hw}_{I_0}^{\leq s_p}(B^{p-1}).$$

Note that we have from Remark 5.17 (i) that

$$\Psi_B(u_{s_p\Lambda_0} \otimes m(B_p) \otimes \tau(b')) = \tilde{\tau}(u_{(s_p-s_{p-1})\Lambda_0} \otimes \Psi_{B^{p-1}}(u_{s_{p-1}\Lambda_0} \otimes b')).$$

Put $t = \tau^{-1}(0)$. Since $\Psi_{B^{p-1}}$ is a $U'_q(\mathfrak{g})$ -crystal isomorphism, we have from the induction hypothesis that

$$\begin{aligned} \text{wt } \Psi_B(u_{s_p\Lambda_0} \otimes m(B_p) \otimes \tau(b')) &= \tau(\text{aff} \circ \text{wt}(b') + s_p\Lambda_0 - (D(b') - C')\delta) \\ &= \text{aff} \circ \text{wt}(\tau(b')) + s_p\Lambda_{\tau(0)} - (D(b') - (\text{wt}(b'), \varpi_t) - C')\delta \\ &= \text{aff} \circ \text{wt}(\tau(b')) + s_p\Lambda_{\tau(0)} - (D(b') - \langle \text{wt}(b'), \varpi_t^\vee \rangle - C')\delta \end{aligned}$$

for some global constant C' , where the second equality follows since $\tau = (\tau^t)^{-1} = w_t^{-1}t_{-\varpi_t}$ by Lemma 2.1 (iii), and the third one follows from (3). On the other hand, we have from Lemma 7.8 that

$$D(m(B_p) \otimes \tau(b')) = D(b') - \langle \text{wt}(b'), \varpi_t^\vee \rangle + C''$$

for some global constant C'' . Hence the equation (15) holds for every $b \in \text{hw}_{I_0}^{\leq s_p}(B)$, and the theorem is proved from the claim. \square

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